Analysis SEP 2026: The Banach Fixed-Point Theorem / Contraction Mapping Principle

1 Warm-up

1. Today, we will be working a lot with complete metric spaces.
   (a) Define a complete metric space.
   (b) Identify some complete metric spaces (along with their associated metrics!)
   (c) Recall that every Banach space (i.e., complete normed vector spaces) is a complete metric space, but the converse is false. Can you come up with some examples of complete metric spaces that are not Banach spaces?
2. The focus of today's session will be on the Banach fixed-point theorem / contraction mapping principle. (These names both refer to the same theorem.)
   (a) Give the full statement of the theorem.
   (b) State a corollary of the theorem that involves iterates of a self-map on a complete metric space.
3. Suppose Helen the Pug is hanging out in the 9th floor lounge and is carrying a map of the city of Madison. Word unfortunately spreads that she is (illegally) in the building, and campus police arrives to escort her out; unfortunately, along the way, she drops the map and it lands on the floor of the lounge. Prove that there exists unique point on the map which sits exactly above the corresponding point in the lounge.
   Hint: Let $M \subset \mathbb{R}^2$ denote the city of Madison (say, represented by the appropriate longitudes and latitudes, if that makes your life easier). How should we define a map $F : M \to M$ so that the Banach fixed-point theorem applies?

2 Exercises

Problem 8.23.3 Consider the space $C([0,10])$ of continuous functions on $[0,10]$, and for a given number L consider the metric $d_L(f,g) = \max_{x\in[0,10]} e^{-Lx}|f(x) - g(x)|$.

(i) Argue that $C([0,10])$ with the metric d_L is a complete metric space.

(ii) Show that there is a unique function which is continuous on $[0,10]$ and satisfies the equation
$$f(x) = -15 + \cos(x) \int_0^x e^{e^{tx}} f(t)\, dt,$$
for all $x \in [0,10]$.
Hint: Apply the contraction mapping principle using the metric d_L for suitable L.

Problem 1.24.1 Prove that there are two functions $f_1, f_2 \in C[0,1]$ that solve the following system of equations for all $x \in [0,1]$.

$$20f_1(x) + 3f_2(x) = \sin x + \int_0^1 \sin(xt)\sin f_1(t)\,dt$$

$$-f_1(x) + 10f_2(x) = \cos x - \int_0^{1/2} \cos(xt)\cos f_2(t)\,dt?$$

Problem 8.24.3 Let $u : \mathbb{R} \to \mathbb{R}$ be a C^1-function, such that $|u'(x)| \leq a < 1/2$ for all $x \in \mathbb{R}$. Define a C^1-function $g : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g(x,y) = (x + u(2y), y + u(x)).$$

Show that g is surjective.

Problem 8.20.1 Let E be a C^2 real function near the origin of $\mathbb{R}^2$. Suppose that E as well as all its first-order and second-order partial derivatives vanish at the origin.

(a) Show that there exist a positive constant b and a continuous function g on $[0,b]$ satisfying

$$xg(x) = E(x, g(x)), \quad x \in [0,b], \quad g(0) = 0.$$

(b) Show that any function g in (a) is in $C^2([0,s])$ if $s > 0$ is sufficiently small. (If it helps, you may further assume that g is in $C^1([0,s])$.)

3 Homework

Applications of the Banach fixed-point theorem are frequently asked on the qualifying exam.

- Go through all of the old qualifying exams and identify which problems beyond the ones on this list require applying this theorem in some way.
- From the list you made, try to prove as many of the exercises as you can.